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On the Existence of Extremal Projections

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1. INTRODUCTION

A central problem in approximation theory can be described thus: a linear subspace Y is prescribed in a normed linear space X. For each $x \in X$ it is desired to estimate the quantity

$$\operatorname{dist}(x, Y) := \inf\{\| x - y \| : y \in Y\}$$

and to determine points $y \in Y$ for which this infimum is attained or nearly attained. A related problem is that of discovering a well-behaved map $P : X \to Y$ such that for each $x \in X$, $\| x - Px \|$ is close to $\operatorname{dist}(x, Y)$. If P is a bounded linear map of X into Y, and if we impose the reasonable requirement

$$\| x - Px \| \leqslant \lambda \cdot \operatorname{dist}(x, Y)$$

for some λ and for all x, then it is clear that $Py = y$ for all $y \in Y$. A bounded linear map P from X into Y is called a projection of X onto Y if $Py = y$ for all $y \in Y$. For such a map it is readily proved that for all $x \in X$

$$\| x - Px \| \leqslant \| id_X - P \| \cdot \operatorname{dist}(x, Y).$$

It is therefore of some importance to determine projections P of X onto Y for which $\| id_X - P \|$ is small, since these have the most favorable approximation properties. A closely related problem is that of making $\| P \|$ small. The following definitions are therefore introduced. If Y is complemented in X, the numbers

$$\mathscr{P}(Y, X) := \inf\{\| P \| : P \text{ projection of } X \text{ onto } Y\}$$
$$\mathscr{P}^*(Y, X) := \inf\{\| id_X - P \| : P \text{ projection of } X \text{ onto } Y\}$$

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are called the *relative projection constant*, resp. the *relative co-projection constant*, of Y with respect to X. These constants are called *exact* if there exists a projection of X onto Y for which the corresponding infimum is attained. Sometimes it is of interest to consider only special projections. The following terminology is therefore introduced. If S is a set of projections from X onto Y and if P_0 is an element of S such that $\| P_0 \| \leqslant \| P \|$ for each P in S, then P_0 is termed a *minimal* element of S. If on the other hand, $\| id_X - P_0 \| \leqslant \| id_X - P \|$ for each P in S, then P_0 is termed a *co-minimal* element of S. These notions can, of course, be translated into the more familiar language of the theory of best approximation. Recall that an arbitrary subset Y of a normed linear space X is termed an *existence subset* of X if every element x of X has a *best approximation* in Y, i.e., an element y of Y such that $\| x - y \| = \text{dist}(x, Y) := \inf\{\| x - y \| : y \in Y\}$. If we let then for two normed linear spaces X and Y, $B(X, Y)$ denote the normed linear space of all bounded linear maps of X into Y with the usual norm

$$\| L \| := \sup\{\| Lx \| : x \in X \quad \text{and} \quad \| x \| \leqslant 1\},$$

we have: If Y is a linear subspace of a normed linear space X and if S is a set of projections from X onto Y, then the minimal elements of S are just the best approximations in S of $0 \in B(X, X)$ and the cominimal elements of S are the best approximations in S of $id_X \in B(X, X)$.

For an exposition of the theory of minimal and cominimal projections the reader is referred to [1].

2. Exactness of Relative Projection and Co-Projection Constants

One of our basic tools in proving the existence of minimal and cominimal projections is the following well-known theorem of Phelps [5]: A $\sigma(X^*, X)$-closed subset of the dual X^* of a normed linear space X is an existence subset of X^*. This theorem becomes applicable to the problem under consideration by the following observation:

If X and Y are normed linear spaces, the space $B(X, Y^*)$ is, under the following canonical map, isometrically isomorphic to the dual of the (not completed) tensor-product $X \otimes_\gamma Y$ of X and Y, equipped with the greatest cross-norm.

$$\kappa : (X \otimes_\gamma Y)^* \to B(X, Y^*)$$

$$((\kappa f)(x))(y) := f(x \otimes y), \qquad f \in (X \otimes_\gamma Y)^*, \qquad x \in X, \qquad y \in Y.$$

Hence $B(X, Y^*)$ "is" a dual space and, as we can easily see, the corresponding weak*-topology on $B(X, Y^*)$ is such that a net $\{L_i : i \in I\}$ converges to zero if and only if $(L_i x)(y) \to 0$ for all $x \in X$ and $y \in Y$. Compare this with Isbell–Semadeni [4] and Ikebe [3].

With this observation one obtains at once the following corollaries of Phelps' theorem.

COROLLARY (J. R. Isbell-Z. Semadeni [4]). *If Y is a complemented linear subspace of a normed linear space X and if Y is isometrically isomorphic to the dual of a normed linear space Z, then $\mathscr{P}(Y, X)$ is exact.*

Proof. The set $\{L \in B(X, Y) : Lx = x \text{ for all } x \in Y\}$ is, under the obvious isometric isomorphism of $B(X, Y)$ onto $B(X, Z^*)$, a weak*-closed subset of $B(X, Z^*)$.

COROLLARY. *If Y is a complemented reflexive linear subspace of a normed linear space X, then both $\mathscr{P}(Y, X)$ and $\mathscr{P}^*(X, Y)$ are exact.*

Proof. If we consider, under the canonical map, X as a subspace of X^{**} and observe that then $Y = Y^{\perp\perp}$, the set $\{L \in B(X, X^{**}) : L[X] = Y$ and $Lx = x$ for all $x \in Y\}$ is a weak*-closed subset of $B(X, X^{**})$.

COROLLARY. *If Y is a linear subspace of a normed linear space X and if $Y^\perp$ is complemented in X^*, then both $\mathscr{P}(Y^\perp, X^*)$ and $\mathscr{P}^*(Y^\perp, X^*)$ are exact.*

Proof. The set $\{L \in B(X^*, X^*) : L$ projection of X^* onto $Y^\perp\}$ is a weak*-closed subset of $B(X^*, X^*)$.

Our next theorem is essentially a device for constructing new spaces with exact relative projection or coprojection constants from old ones.

Let F be a linear subspace of the linear space of all real-valued functions on a set I and let F be normed by a norm N. Assume that, whenever f and g are real-valued functions on I such that $f \in F$ and $|g| \leqslant |f|$, then $g \in F$ and $Ng \leqslant Nf$. For each $i \in I$ let X_i be a normed linear space. Then the set X of all mappings of I into the union of the X_i which have the property, that $x(i) \in X_i$ for $i \in I$ and that the real-valued function defined by $\tilde{x}(i) := \| x(i) \|$ for $i \in I$ is in F, is a normed linear space under the usual vector space operations and the norm $\| x \| := N\tilde{x}$ for $x \in X$. Such a space X is termed a "substitution space" by Day [2, p. 31]. Of particular interest are those substitution spaces where F is one of the spaces $c_0(I)$ or $l_p(I)$, $1 \leqslant p < \infty$. We define maps $\pi_i : X \to X_i$ and $\eta_i : X_i \to X$ by $\pi_i x := x(i)$ and $(\eta_i u)(i) := u$, $(\eta_i u)(j) := 0$ for $i \in I$, $j \in I \sim \{i\}$, and make the extra assumption that all the embeddings η_i are isometries.

THEOREM. *Let X be the substitution space defined above. For each $i \in I$ let Y_i be a linear subspace of X_i and let $Y := \{x \in X : x(i) \in Y_i \text{ for all } i \in I\}$.*

(i) *Y is complemented in X if and only if each Y_i is complemented in X_i and* $\sup\{\mathscr{P}(Y_i, X_i) : i \in I\} < \infty$ (*the last condition can, of course, be replaced by* $\sup\{\mathscr{P}^*(Y_i, X_i) : i \in I\} < \infty$).

(ii) *If Y is complemented in X, then* $\mathscr{P}(Y, X) = \sup\{\mathscr{P}(Y_i, X_i) : i \in I\}$ *and* $\mathscr{P}^*(Y, X) = \sup\{\mathscr{P}^*(Y_i, X_i) : i \in I\}$.

(iii) *If Y is complemented in X, then $\mathscr{P}(Y, X)$ is exact if and only if for each $i \in I$, there exists a projection P_i of X_i onto Y_i such that $\| P_i \| \leqslant \mathscr{P}(Y, X)$, and $\mathscr{P}^*(Y, X)$ is exact if and only if for each $i \in I$, there exists a projection P_i of X_i onto Y_i such that $\| id_{X_i} - P_i \| \leqslant \mathscr{P}^*(Y, X)$.*

(iv) *Y is an existence subset of X if and only if for each $i \in I$, Y_i is an existence subset of X_i.*

Proof. (a) If P is a projection of X onto Y, then for each $i \in I$, $\pi_i P \eta_i$ is a projection of X_i onto Y_i such that $\| P \| \geqslant \| \pi_i P \eta_i \|$ and $\| id_X - P \| \geqslant \| id_{X_i} - \pi_i P \eta_i \|$.

To prove (a), let $i \in I$. If $u \in X_i$, then $P\eta_i u \in Y$, i.e., $\pi_i P \eta_i u \in Y_i$ and if $u \in Y_i$, then $\eta_i u \in Y$, hence $P\eta_i u = \eta_i u$, i.e., $\pi_i P \eta_i u = \pi_i \eta_i u = u$. This shows that $\pi_i P \eta_i$ is a projection of X_i onto Y_i. To complete the proof of (a), we observe that for each $u \in X_i$ such that $\| u \| \leqslant 1$ we have $\| \eta_i u \| = \| u \| \leqslant 1$ and hence

$$\| P \| \geqslant \| P\eta_i u \| \geqslant \| \pi_i P \eta_i u \|$$

and

$$\| id_X - P \| \geqslant \| \eta_i u - P\eta_i u \| \geqslant \| \pi_i \eta_i u - \pi_i P \eta_i u \| = \| u - \pi_i P \eta_i u \|$$

and from this the remaining two inequalities of (a) follow immediately.

(b) If $\lambda \in \mathbb{R}$ is such that for each $i \in I$ there exists a projection P_i of X_i onto Y_i with the property $\| P_i \| \leqslant \lambda$, resp., $\| id_{X_i} - P_i \| \leqslant \lambda$, then the map $P : X \to X$ defined by $P(x)(i) := P_i \pi_i x$ for $x \in X$ and $i \in I$ is a projection of X onto Y with the property $\| P \| \leqslant \lambda$, resp., $\| id_X - P \| \leqslant \lambda$.

If $x \in X$, then for every $i \in I$, $(Px)(i) = P_i \pi_i x \in Y_i$, i.e., $Px \in Y$, and if $x \in Y$, then for every $i \in I$, $\pi_i x \in Y$, i.e., $(Px)(i) = P_i \pi_i x = \pi_i x$ and hence $Px = x$. This shows that P is a projection of X onto Y. If now $\| P_i \| \leqslant \lambda$ for $i \in I$, then $\|(Px)(i)\| = \| P_i \pi_i x \| \leqslant \lambda \| \pi_i x \| = \lambda \| x(i) \|$, i.e., $0 \leqslant \widetilde{Px} \leqslant \lambda \tilde{x}$, i.e., $N\widetilde{Px} \leqslant \lambda \cdot N\tilde{x}$, i.e., $\| Px \| \leqslant \lambda \| x \|$ and this shows that $\| P \| \leqslant \lambda$. If finally $\| id_{X_i} - P_i \| \leqslant \lambda$ for $i \in I$, then $\|(x - Px)(i)\| = \| \pi_i x - P_i \pi_i x \| \leqslant \lambda \| \pi_i x \| = \lambda \| x(i) \|$, i.e.,

$$0 \leqslant \widetilde{x - Px} \leqslant \lambda \tilde{x}, \text{ i.e., } N(\widetilde{x - Px}) \leqslant \lambda \cdot N\tilde{x}, \text{ i.e., } \| x - Px \| \leqslant \lambda \| x \|$$

and this shows that $\| id_X - P \| \leqslant \lambda$.

(a) and (b) imply now obviously (i)–(iii). (iv) is a trivial exercise which was only included for later reference.

The theorem that was just proved sheds some light on the following problem: Whereas, as we have shown earlier, finite-dimensional linear subspaces of normed linear spaces always have exact relative projection and coprojection constants, it is well known that the same is not true for linear subspaces of finite codimension. It was conjectured in [1] that existence subspaces of finite codimension would have exact relative projection constants. While we cannot prove this conjecture, we obtain from our theorem the following:

COROLLARY. *Every existence subspace of finite codimension in c_0 has an exact relative projection and coprojection constant.*

Proof. Let Y be a finite-codimensional linear subspace of c_0. Y is an existence subset of c_0 if and only if every element of $Y^\perp$ attains its norm on the unit ball of c_0. The necessity is in Phelps [6]. To show the sufficiency, we observe that an element of l_1 (considered in the usual way as the dual of c_0) attains its norm on the unit ball of c_0 if and only if it has at most a finite number of nonzero coordinates. Since $Y^\perp$ is finite-dimensional, this implies the existence of an $n \in \mathbb{N}$ such that for every $f \in Y^\perp$, $f_i = 0$ for $i \geqslant n + 1$. Let now

$$X_1 := \{x \in c_0 : x_i = 0 \text{ for } i \leqslant n\} \text{ and } X_2 := \{x \in c_0 : x_i = 0 \text{ for } i \geqslant n + 1\}.$$

Then c_0 is the c_0-sum of X_1 and X_2 and $Y = (Y \cap X_1) \oplus (Y \cap X_2)$. Since $Y_1 := Y \cap X_1 = X_1$ is an existence subspace of X_1 with an exact relative projection and coprojection constant and since the same holds for the linear subspace $Y_2 = Y \cap X_2$ of X_2, the sufficiency as well as the corollary follow directly from our last theorem.

3. MINIMAL ORTHOGONAL PROJECTIONS

The concept of *orthogonal* projection comes from classical approximation theory and is defined as follows. Let Y be a linear subspace of the normed linear space $X = C([a, b)]$. Let P be a projection of X onto Y. If there exists a monotone function σ of variation 1 such that

$$\int_a^b [x(t) - (Px)(t)]\, y(t)\, d\sigma(t) = 0 \qquad (y \in Y, \quad x \in X), \tag{2}$$

then P is termed an *orthogonal* projection. In the most common cases, σ has the property that the induced pseudo-inner-product

$$\langle x, z\rangle = \int_a^b x(t)\, z(t)\, d\sigma(t)$$

is a genuine inner product on Y. Thus $\langle y, y\rangle > 0$ for all $y \in Y \sim \{0\}$. In this case, if Y is finite-dimensional, then there exists an orthonormal base $\{y_1, ..., y_n\}$ for Y, and P is given by the expression

$$Px = \sum_{i=1}^{n} \langle x, y_i\rangle\, y_i\,.$$

A natural question that arises from these considerations is whether the function σ can be chosen in such a manner as to make $\| P \|$ a minimum. It is proved below that this is always possible if Y is finite-dimensional.

In a general space $C(T)$, with T compact Hausdorff, the condition (1) is replaced by the requirement that there exist a nonnegative $f \in X^* \sim \{0\}$ such that

$$f((x - Px) \cdot y) = 0 \qquad (y \in Y, x \in X) \tag{2}$$

THEOREM. *Let Y be a finite-dimensional linear subspace of a normed linear space X. Let there be prescribed a $\sigma(X^*, X)$-compact subset K of X^* and a set Λ of bounded linear maps from X into X. Then the set Π of all projections P from X onto Y such that $f \circ L \circ (id_X - P) = 0$ for some $f \in K$ and for all $L \in \Lambda$ is a weak*-closed subset of $B(X, Y)$, and hence contains a minimal element.*

Proof. Consider a net $\{P_i : i \in I\}$ in Π and suppose that $P_i \to P \in B(X, Y)$ in the weak*-topology of $B(X, Y)$. It is immediately seen that P is a projection of X onto Y, and it remains to prove that there exists an $f \in K$ such that $f \circ L \circ (id_X - P) = 0$ for all $L \in \Lambda$. For each i there exists $f_i \in K$ such that $f_i \circ L \circ (id_X - P_i) = 0$ for all $L \in \Lambda$. By passing to a suitable subnet we can assume that $f_i \to f \in K$ in the topology $\sigma(X^*, X)$. Fixing $x \in X$ and $L \in \Lambda$, we have

$$\begin{aligned} | fLx - fLPx | &\leqslant | fLx - f_iLx | \\ &\quad + | f_iLP_ix - f_iLPx | + | f_iLPx - fLPx | \leqslant |(f - f_i)\, Lx | \\ &\quad + (\sup_i \| f_i \|)\, \| L \|\, \| P_ix - Px \| + |(f_i - f)\, LPx |. \end{aligned}$$

Since Y is finite-dimensional, $\| P_ix - Px \| \to 0$. Since $f_i \in K$, $\sup_i \| f_i \| < \infty$. Thus the inequalities above establish that $f \circ L \circ (id_X - P) = 0$.

COROLLARY. *The set of all orthogonal projections from $C(T)$ onto a fixed finite-dimensional linear subspace Y is a weak*-closed subset of $B(C(T), Y)$, and hence contains a minimal element.*

Proof. In the preceding theorem, let $X = C(T)$ and let $K = \{f \in X^* : f \geq 0$ and $\|f\| = 1\}$. For each $y \in Y$ let L_y be the multiplication operator defined by $L_y x = y \cdot x$, $x \in X$, and let $\Lambda := \{L_y : y \in Y\}$. The set Π in the theorem is then exactly the set of all orthogonal projections from X onto Y.

The family of projections that have been termed "orthogonal" is rather large and includes projections which are not conventionally regarded as orthogonal, e.g., all Lagrange interpolating projections [1]. The following example shows, however, that one cannot, in general, expect to obtain the existence of minimal orthogonal projections from our last theorem if the notion of orthogonality is narrowed down to a classical one.

EXAMPLE. Let $X := C([0, 1])$, $y_1(t) := 1$ and $y_2(t) := t^2$ for $t \in [0, 1]$ and let $Y := \operatorname{span}\{y_1, y_2\}$. Let Π be the set of all projections of X onto Y that have a representation

$$\text{(i)} \qquad Px = \sum_{i=1}^{2} f(xy_i)\, y_i, \qquad x \in X,$$

for some $f \in X^*$. If (i) holds, then necessarily

$$\text{(ii)} \qquad f(y_i y_j) = \delta_{ij}, \qquad i, j \in \{1, 2\},$$

and conversely, whenever $f \in X^*$ satisfies (ii), then the map defined by (i) is a projection of X onto Y. Furthermore, if $f \in X^*$ satisfies (ii),

$$\text{(iii)} \qquad \langle x, z \rangle := \sum_{i=1}^{2} f(xy_i)\, f(zy_i), \qquad x, z \in X,$$

defines a pseudo inner product for X which is an inner product for Y with respect to which the map defined by (i) is orthogonal,

$$\text{(iv)} \qquad \langle x - Px, y \rangle = 0, \qquad x \in X, \quad y \in Y.$$

We do not know whether the set Π defined above contains a minimal element. We can show, however, that Π is not a weak*-closed subset of $B(X, Y)$: Define a sequence $\{f_n : n \in \mathbb{N}\}$ in X^* by $f_n(x) := n \cdot x(n^{-1}) + g_n(x)$ $x \in X$, $n \in \mathbb{N}$, where $\{g_n : n \in \mathbb{N}\}$ is a $\sigma(X^*, X)$-convergent sequence in X^* such that $f_n(y_i y_j) = \delta_{ij}$, $n \in \mathbb{N}$, $i, j \in \{1, 2\}$ (the existence of such a sequence $\{g_n : n \in \mathbb{N}\}$ is easily proved). Then, if g denotes the $\sigma(X^*, X)$-limit of $\{g_n : n \in \mathbb{N}\}$, the projections corresponding to the functionals f_n by (i) converge in the weak*-topology of $B(X, Y)$ to the projection P of X onto Y which is defined by

$Px : \sum_{i=1}^{2} h_i(x)\, y_i$ with $h_1(x) := x(0) + g(xy_1)$, $h_2(x) := g(xy_2)$, $x \in X$. This projection P, however, is not in Π, as can be seen from the following considerations. If $P \in \Pi$, then there exists $f \in X^*$ such that $h_i(x) = f(xy_i)$, $i \in \{1, 2\}$, $x \in X$. Since $y_1 \in \{x \in X : x(0) = 0\}$ and since this is, by the Stone–Weierstrass theorem, the closure in X of the subalgebra $\{x \cdot y_2 : x \in X\}$, there exists a sequence $\{x_n : n \in \mathbb{N}\}$ in X such that $x_n \cdot y_2 \to y_1$. Then the following contradiction arises: $g(y_1) = \lim_n g(x_n y_2) = \lim_n h_2(x_n) = \lim_n f(x_n y_2) = f(y_1) = h_1(1) = 1 + g(y_1)$.

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